Combinatorial Sets of Reals, III

Independence: Spectrum and Genericity

Vera Fischer

University of Vienna

Jan 28–Feb 4, 2023

Winter School in Abstract Analysis

Section Set Theory & Topology

Definition: Spectrum of Independence

$\mathfrak{sp}(\mathfrak{i}) = \{|\mathscr{A}| : \mathscr{A}$ is a max. ind. family$\}$

Theorem (F., Shelah)

Assume *CH*. Let λ be a regular uncountable cardinal. Then

$$V^{\mathbb{S}_\lambda} \vDash \mathfrak{sp}(\mathfrak{i}) = \{\aleph_1, \lambda\}.$$

No intermediate cardinalities

Lemma

In the above extension there are no m.i.f. of size κ, for $\aleph_1 < \kappa < \lambda$.

$\mathscr{A}$-diagonalization filters

Let $\mathscr{A}$ be an independent family. A filter $\mathscr{U}$ is said to be an $\mathscr{A}$-diagonalization filter if

$$\forall F \in \mathscr{U} \forall h \in \mathrm{FF}(\mathscr{A})(|F \cap \mathscr{A}^h| = \omega)$$

and is maximal with respect to the above property.

Lemma

Suppose $\mathscr{U}$ is a $\mathscr{A}$-diagonalization filter, G is $\mathbb{M}(\mathscr{U})$-generic and

$$x_G = \bigcup\{s : \exists F(s,F) \in G\}.$$

Then:

1. $\mathscr{A} \cup \{x_G\}$ is independent
2. If $y \in ([\omega]^\omega \backslash \mathscr{A}) \cap V$ is such that

$$\mathscr{A} \cup \{y\}$$

   is independent, then $\mathscr{A} \cup \{x_G, y\}$ is not independent.

Proof (1):

For $h \in \mathrm{FF}(\mathscr{A})$ and $n \in \omega$, the sets

- $D_{h,n} := \{(s,F) \in \mathbb{M}(\mathscr{U}) : |s \cap \mathscr{A}^h| > n\}$, and
- $E_{h,n} := \{(s,F) : |(\min F \setminus \max s) \cap \mathscr{A}^h| > n\}$

are dense, and so $\mathscr{A}^h \cap x_G$, and $\mathscr{A}^h \setminus x_G$ are infinite.

Proof (2):

Fix y such that $\mathscr{A} \cup \{y\}$ is independent.

1. If $y \in \mathscr{U}$, then $x_G \subseteq^* y$ and so $x_G \setminus y$ is finite.
2. If $y \notin \mathscr{U}$, then
   - either there is $F \in \mathscr{U}$ such that $F \cap y$ is finite, and so $x_G \cap y$ is finite,
   - or there are $F \in \mathscr{U}$, $h \in \mathrm{FF}(\mathscr{A})$ s.t. $F \cap y \cap \mathscr{A}^h = \emptyset$, in which case $x_G \cap y \cap \mathscr{A}^h$ is finite.
3. Thus in either case $\mathscr{A} \cup \{x_G, y\}$ is not independent.

□

Corollary

Let κ be a regular uncountable cardinal. Then consistently

$$\aleph_1 < \mathfrak{i} = \kappa < \mathfrak{c}.$$

Proof:

Let $\lambda > \kappa$ be the desired size of the continuum. The ordinal product $\gamma^* = \lambda \cdot \kappa$ contains an unbounded subset $\mathscr{I}$ of cardinality κ. Consider a finite support iteration of length γ^* such that along $\mathscr{I}$ we

- recursively generate a max. independent family of cardinality κ,
- as well as a scale of length κ,

and along $\gamma^* \backslash \mathscr{I}$, we add Cohen reals. Then in the final generic extension

$$\aleph_1 < \mathfrak{d} = \kappa \leq \mathfrak{i} \leq \kappa < \mathfrak{c} = \lambda.$$

□

Theorem (F., Shelah, 2019)

Assume *GCH*. Let $\kappa_1 < \cdots < \kappa_n$ be regular uncountable cardinals. There is a ccc generic extension in which $\{\kappa_i\}_{i=1}^n \subseteq \mathfrak{sp}(\mathfrak{i})$.

Proof:

Consider a finite support iteration of length γ^*, where γ^* is the ordinal product $\kappa_n \cdot \kappa_{n-1} \cdots \kappa_1$ and elaborate on the previous idea. □

Ultrapowers

Let κ a measurable and let $\mathscr{D} \subseteq \mathscr{P}(\kappa)$ be a κ-complete ultrafilter. Let $\mathbb{P}$ be a p.o. Then $\mathbb{P}^\kappa/\mathscr{D}$ consists of all equivalence classes

$$[f] = \{g \in {}^\kappa\mathbb{P} : \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in \mathscr{D}\}$$

and is supplied with the p.o. relation $[f] \leq [q]$ iff

$$\{\alpha \in \kappa : f(\alpha) \leq_{\mathbb{P}} g(\alpha)\} \in \mathscr{D}.$$

We can identify each $p \in \mathbb{P}$ with

$$[p] = [f_p],$$

where $f_p(\alpha) = p$ for each $\alpha \in \kappa$ and so we can assume $\mathbb{P} \subseteq \mathbb{P}^\kappa/\mathscr{D}$.

Lemma

1. If $\mathbb{P}$ is ccc, then $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathscr{D}$.
2. If $\mathbb{P}$ has the countable chain condition, then so does $\mathbb{P}^\kappa/\mathscr{D}$.

Lemma

If $\mathscr{A}$ be a $\mathbb{P}$-name for an independent family of cardinality $\geq \kappa$. Then

$$\Vdash_{\mathbb{P}^\kappa/\mathscr{D}} \mathscr{A} \text{ is not maximal.}$$

Theorem (F., Shelah, 2019)

Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ be measurable witnessed by κ_i-complete ultrafilters $\mathscr{D}_i \subseteq \mathscr{P}(\kappa_i)$. There is a ccc generic extension in which

$$\{\kappa_i\}_{i=1}^n = \mathfrak{sp}(\mathfrak{i}).$$

Proof/Idea:

Let $\gamma^* = \kappa_n \cdot \kappa_{n-1} \cdots \kappa_1$ and for each $j \in \{1, \cdots, k\}$ fix $\mathscr{I}_j \subseteq \gamma^*$ unbounded, of cardinality κ_j. Along each $\mathscr{I}_j$

- iteratively generate a max. ind. family of cardinality κ_j, and
- for unboundedly many $\alpha \in \mathscr{I}_j$ take the ultrapower $\mathbb{P}_\alpha^{\kappa_j} / \mathscr{D}_j$.

□

Proof:

More precisely, take $\mathscr{I}_j \subseteq \gamma^*$ for $j = 1, \cdots, n$ so that:

- $\mathscr{I}_j$ consists of successor ordinals, $|\mathscr{I}_j| = \kappa_j$
- $\mathscr{I}_j \cap \text{Even}$ and $\mathscr{I}_j \cap \text{Odd}$ are unbounded in γ^*, and
- $\{\mathscr{I}_j\}_{j=1}^{j=n}$ are pairwise disjoint.

Define a finite support iteration of length γ^* as follows. Fix $\alpha < \gamma$ and suppose for each $k \in \{1, \cdots, n\}$ a sequence of reals

$$\langle r^k_\gamma : \gamma \in \mathscr{I}_k \cap \text{Even}, \gamma < \alpha \rangle$$

has been defined such that

- $\mathscr{A}^k_\alpha = \bigcup \{r^k_\gamma : \gamma \in \mathscr{I}_k \cap \text{Even} \cap \alpha\}$ is independent, and
- for each $\gamma \in \mathscr{I}_k \cap \text{Even}$, r^k_γ diagonalizes $\mathscr{A}^k_\gamma$ over $V^{\mathbb{P}_\gamma}$.

Proof (cnt'd):

Proceed as follows.

1. If $\alpha \in \mathscr{I}_k \cap \text{Even}$ for some $k \in \{1, \cdots, n\}$ then
   - choose an $\mathscr{A}^k_\alpha$-diagonalizing filter $\mathscr{U}_\alpha$ in $V^{\mathbb{P}_\alpha}$,
   - take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for $\mathbb{M}(\mathscr{U}_\alpha)$, and
   - r^k_α to be the associated Mathias generic real.
2. If $\alpha \in \mathscr{I}_k \cap \text{Odd}$ for some $k \in \{1, \cdots, n\}$, then
   - $\alpha = \beta + 1$ and so we take
   - $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\beta$-name for the quotient of $\mathbb{P}^{\kappa_k}_\beta / \mathscr{D}_k$ and $\mathbb{P}_\beta$.
   - Thus, in particular $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\alpha$.
3. If $\alpha \notin \bigcup_{k=1}^n \mathscr{I}_k$ take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the Cohen poset. □

Question:

- Can we have a precise evaluation of the spectrum, without the assumption of measurables?
- Can we adjoin via forcing a maximal independent family of cardinality $\aleph_\omega$?

Lemma

Let $\mathscr{A}$ be an independent family, $\mathscr{U}$ a $\mathscr{A}$-diagonalization filter. Let $n > 1$ and for each $i \in n$ let $\mathscr{U}_i = \mathscr{U}$. Let

$$G = \prod_{i \in n} G_i \text{ be } \mathbb{P} = \prod_{i \in n} \mathbb{M}(\mathscr{U}_i)\text{-generic filter}$$

and for each $i \in n$ let $x_i = x_{G_i}$. Then in $V[G]$:

1. $\mathscr{A} \cup \{x_i\}_{i \in n}$ is independent;
2. if $y \in (V \backslash \mathscr{A}) \cap [\omega]^\omega$ be such that

$$\mathscr{A} \cup \{y\} \text{ is independent,}$$

   then for each $i \in n$, the family $\mathscr{A} \cup \{y, x_i\}$ is not independent.

Proof

Item (2) holds, since each x_i is a diagonalization real.

To prove item (1):

- fix $h \in \mathrm{FF}(\mathscr{A})$ and an arbitrary $j : n \to 2$;
- for each $n \in \omega$, we will show that the set

$$D_{h,j,n} = \{\langle (t_i, H_i)\rangle_{i\in n} : \exists i^* > n(i^* \in \bigcap t_i^{j(i)} \cap \mathscr{A}^h)\}$$

  is dense in $\mathbb{P}$, where $t_i^0 = t$, $t_i^1 = \min H_i \setminus t_i$. Thus, if $p \in D_{h,j,n}$ then

$$p \Vdash i^* \in \bigcap_{i\in n} x_i^{j(j)} \cap \mathscr{A}^h,$$

  where $x_i^0 = x_i$ and $x_i^1 = \omega \setminus x_i$.

Proof cnt'd:

- Let $\bar{p} = \langle (s_i, F_i) \rangle_{i \in n} \in \mathbb{P}$. Let $I = \{i \in n : j(i) = 0\}$ and $J = n \backslash I$.
- Thus, for each $i \in I$, $s_i^{j(i)} = s_i$ and for each $i \in J$, $s_i^{j(i)} = \omega \backslash s_i$.
- Since $\mathscr{U}$ is $\mathscr{A}$-diagonalization,

$$\bigcap_{i \in I} F_i \cap \mathscr{A}^h$$

is infinite and so there is

$$i^* \in \bigcap_{i \in I} F_i \cap \mathscr{A}^h,$$

which is strictly bigger than n and the maximum of s_i for all $i \in n$.

Proof cnt'd:

Then:

1. if $i \in I$, $(s_i \cup \{i^*\}, F_i \backslash (i^* + 1)) \leq (s_i, F_i)$ and forces $i^* \in x_i \cap \mathscr{A}^h$;
2. if $i \in J$, $(s_i, F_i \backslash (i^* + 1)) \leq (s_i, F_i)$ and forces $i^* \in (\omega \backslash x_i) \cap \mathscr{A}^h$.

Let $\bar{q} = \langle q_i \rangle_{i \in n}$ where

$$q_i = (s_i \cup \{i^*\}, F_i \backslash (i^* + 1)) \text{ for } i \in I,\ q_i = (s_i, F_i \backslash (i^* + 1)) \text{ for } i \in J.$$

Then $\bar{q} \leq \bar{p}$ and $\bar{q} \in D_{h,j,n}$. In particular,

$$\bar{q} \Vdash i^* \in \bigcap_{i \in n} x_i^{j(i)} \cap \mathscr{A}^h.$$

□

Proof cnt'd:

Then:

1. if $i \in I$, $(s_i \cup \{i^*\}, F_i \backslash (i^* + 1)) \leq (s_i, F_i)$ and forces $i^* \in x_i \cap \mathscr{A}^h$;
2. if $i \in J$, $(s_i, F_i \backslash (i^* + 1)) \leq (s_i, F_i)$ and forces $i^* \in (\omega \backslash x_i) \cap \mathscr{A}^h$.

Let $\bar{q} = \langle q_i \rangle_{i \in n}$ where

$$q_i = (s_i \cup \{i^*\}, F_i \backslash (i^* + 1)) \text{ for } i \in I, \; q_i = (s_i, F_i \backslash (i^* + 1)) \text{ for } i \in J.$$

Then $\bar{q} \leq \bar{p}$ and $\bar{q} \in D_{h,j,n}$. In particular,

$$\bar{q} \Vdash i^* \in \bigcap_{i \in n} x_i^{j(i)} \cap \mathscr{A}^h.$$

□

Theorem (F., Shelah)

(GCH) Let θ be an uncountable cardinal. Then, there is a ccc poset, which adjoins a maximal independent family of cardinality θ.

Remark

In particular, there is a ccc poset adjoining a maximal independent family of cardinality $\aleph_\omega$.

Definition

Fix $\sigma \leq \theta \leq \lambda$, where:

- σ is regular uncountable (the intended value of $\mathfrak{i}$),
- λ is of uncountable cofinality (the intended value of $\mathfrak{c}$).
- Let $S \subseteq \theta^{<\sigma}$ be a well-prunded θ-splitting tree of height σ.
- For each $\alpha < \sigma$, let S_α be the α-th splitting level of S.

Recursively define a finite support iteration

$$\mathbb{P}_S = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \sigma, \beta < \sigma \rangle$$

of length σ such that for each α, in $V^{\mathbb{P}_\alpha}$ we have

$$\mathbb{Q}_\alpha = \prod_{\eta \in S_\alpha} \mathbb{Q}_\eta$$

where $\mathbb{Q}_\eta$ is Mathias forcing for an appropriate diagonalization filter.

More precisely:

- Let $\mathbb{P}_0 = \{\emptyset\}$, $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for the trivial poset.
- Let $\mathscr{A}_0 = \emptyset$ and let $\mathscr{U}_0$ be an arbitrary ultrafilter extending the Fréchet filter. Thus $\mathscr{U}_0$ is $\mathscr{A}_0$-diagonalizing.
- For each $\eta \in S_1 = \mathrm{succ}_S(\emptyset)$, let $\mathscr{U}_\eta = \mathscr{U}_0$ and let
$$\mathbb{Q}_1 = \prod_{\eta \in S_1} \mathbb{M}(\mathscr{U}_\eta)$$
with finite supports.
- In $V^{\mathbb{P}_1 * \dot{\mathbb{Q}}_1}$ for each $\eta \in S_1$ let a_η be the $\mathbb{M}(\mathscr{U}_\eta)$-generic real.

- Suppose $\alpha \geq 2$ and in $V^{\mathbb{P}_\alpha}$ for all $\eta \in S_\alpha$,

$$\mathscr{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}_S(\eta \restriction \xi), \xi < \alpha\}$$

  is independent.

- For each $\eta \in S_\alpha$, let $\mathscr{U}_\eta$ be a $\mathscr{A}_\eta$-diagonalization filter and let

$$\mathbb{Q}_\alpha = \prod_{\eta \in S_\alpha} \mathbb{M}(\mathscr{U}_\eta)$$

  with finite supports.

- In $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha}$ for each $\eta \in S_\alpha$ let a_η be the $\mathbb{M}(\mathscr{U}_\eta)$-generic real.

Lemma

In $V^{\mathbb{P}_S}$ for each branch $\eta \in [S]$ the family

$$\mathscr{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}(\eta \restriction \xi), \xi < \alpha\}$$

is a maximal independent family of cardinality θ.

Proof:

Maximality follows from the diagonalization properties and the fact that the length of the iteration is of uncountable cofinality. □

Corollary (F., Shelah)

There is a ccc forcing notion adjoining a maximal independent family $\mathscr{A}$ such that

$$|\mathscr{A}| = \aleph_\omega.$$

Proof:

Use an $\aleph_\omega$-splitting tree of height ω_1. □

Theorem (F., Shelah, 2022)

Assume GCH.Let σ be a regular uncountable cardinal, λ a cardinal of uncountable cofinality such that $\sigma \leq \lambda$. Let

$$\Theta_1 \subseteq [\sigma, \lambda]$$

be such that

$$\sigma = \min \Theta_1, \max \Theta_1 = \lambda.$$

Then there is a ccc generic extension in which

$$\Theta_1 \subseteq \mathfrak{sp}(\mathfrak{i}).$$

Proof:

Let $\mathbf{m} = \langle S_\theta : \theta \in \Theta_1 \rangle$ be a sequence of pairwise disjoint trees such that for each $\theta \in \Theta_1$, S_θ is a θ-splitting tree of height σ.

Let $\alpha < \sigma$.

- For each $\theta \in \Theta_1$ let $S_{\theta,\alpha}$ denote the α-th splitting level of S_θ and
- Let $S_{\mathbf{m},\alpha} = \bigcup_{\theta \in \Theta_1} S_{\theta,\alpha}$.

Proof cnt'd:

We will define a finite support iteration

$$\mathbb{P}_{\mathbf{m}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \sigma, \beta < \sigma \rangle$$

where for each $\beta < \sigma$ in $V^{\mathbb{P}_\beta}$,

$$\mathbb{Q}_\beta = \prod_{\eta \in S_{\mathbf{m},\beta}} \mathbb{Q}_\eta$$

with finite supports and $\mathbb{Q}_\eta$ is Mathias forcing for an appropriate diagonalization filter adjoining a diagonalization real a_η.

More precisely:

- Let $\mathbb{P}_0 = \{\emptyset\}$, $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for the trivial poset.
- Let $\mathscr{A}_0 = \emptyset$ and let $\mathscr{U}_0$ be an arbitrary ultrafilter extending the Fréchet filter. Thus $\mathscr{U}_0$ is $\mathscr{A}_0$-diagonalizing.
- For each $\eta \in S_{\mathbf{m},1}$ let $\mathscr{U}_\eta = \mathscr{U}_0$ and let

$$\mathbb{Q}_1 = \prod_{\eta \in S_{\mathbf{m},1}} \mathbb{M}(\mathscr{U}_\eta)$$

  with finite supports.
- In $V^{\mathbb{P}_1 * \dot{\mathbb{Q}}_1}$ for each $\eta \in S_{\mathbf{m},1}$ let a_η be the $\mathbb{M}(\mathscr{U}_\eta)$-generic real.

Suppose $\alpha \geq 2$, $\theta \in \Theta_1$, $\eta \in S_{\theta,\alpha}$ and

$$\Vdash_{\mathbb{P}_\alpha} \mathscr{A}_\eta = \{a_\nu : \nu \in \text{succ}_{S_\theta}(\eta \restriction \xi), \xi < \alpha\} \text{ is independent.}$$

Then in $V^{\mathbb{P}_\alpha}$, take $\mathscr{U}_\eta$ to be a $\mathscr{A}_\eta$-diagonalization filter and

$$\mathbb{Q}_\eta = \mathbb{M}(\mathscr{U}_\eta).$$

With this the definition of the forcing notion is complete.

Lemma

In $V^{\mathbb{P}_{\mathbf{m}}}$ for each branch $\eta \in [S_\theta] = S_{\theta,\sigma}$, $\theta \in \Theta_1$ the family

$$\mathscr{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}_{S_\theta}(\eta \restriction \xi), \xi < \sigma\}$$

is maximal independent of cardinality θ. Thus,

$$V^{\mathbb{P}_{\mathbf{m}}} \models \Theta_1 \subseteq \mathfrak{sp}(\mathfrak{i}).$$

Proof:

Diagonalization. □

Theorem (F., Shelah)

- For any finite set $C \subseteq \{\aleph_n\}_{n\in\omega\setminus 1}$, consistently
$$\mathrm{sp}(\mathfrak{i}) = C.$$
- For any infinite $C \subseteq \{\aleph_n\}_{n\in\omega\setminus 1}$ and $\lambda > \aleph_\omega$ of uncountable cofinality, consistently
$$\mathrm{sp}(\mathfrak{i}) = C \cup \{\aleph_\omega, \mathfrak{c} = \lambda\}.$$

Comment

Excluding values is an isomorphism of names argument, essentially a counting argument, relying on specific properties of the forcing construction.

Question:

- Is it consistent that $\mathfrak{i} = \aleph_\omega$?
- Is $\mathfrak{sp}(\mathfrak{i})$ closed with respect to singular limits of countable cofinality?

... and once again Maximality

$\forall X \in [\omega]^\omega \setminus \mathscr{A} \exists h \in \mathrm{FF}(\mathscr{A})$ such that $\mathscr{A}^h \cap X$ or $\mathscr{A}^h \setminus X$ is finite.

Dense maximality

Let $\mathscr{A}$ be an independent family. Then $\mathscr{A}$ is said to be densely maximal if for each $X \in [\omega]^\omega \setminus \mathscr{A}$ and every $h \in \mathrm{FF}(\mathscr{A})$ there is $h' \in \mathrm{FF}(\mathscr{A})$ such that $h' \supseteq h$ and $\mathscr{A}^{h'} \cap X$ or $\mathscr{A}^{h'} \setminus X$ is finite.

Remark

Thus, $\mathscr{A}$ is densely maximal if for each $X \in [\omega]^\omega \setminus \mathscr{A}$ the set of $h \in \mathrm{FF}(\mathscr{A})$ such that X does not split $\mathscr{A}$ is dense in $\mathrm{FF}(\mathscr{A})$.

Density filter

Let $\mathscr{A}$ be an independent family. Then

$$\text{fil}(\mathscr{A}) = \{Y \in [\omega]^\omega : \forall h \in \text{FF}(\mathscr{A}) \exists h' \in \text{FF}(\mathscr{A}) \text{ s.t. } h' \supseteq h \text{ and } \mathscr{A}^{h'} \subseteq Y\}$$

is referred to as the density filter of $\mathscr{A}$.

Lemma

A family $\mathscr{A} \subseteq [\omega]^\omega$ is densely maximal if and only if

$$P(\omega) = \text{fil}(\mathscr{A}) \cup \langle \omega \backslash \mathscr{A}^g \mid g \in \text{FF}(\mathscr{A}) \rangle_{\text{dn}}.$$

Definition: Ramsey filter

A p-filter $\mathscr{F}$ is said to be Ramsey if for every partition $\mathscr{E} = \{E_n\}_{n\in\omega}$ of ω into finite sets, there is a set C in $\mathscr{F}$ such that $|C \cap E_n| \leq 1$ for each n.

Definition: Selective independence

A densely maximal independent family $\mathscr{A}$ such that $\text{fil}(\mathscr{A})$ is Ramsey is said to be selective.

Theorem (Shelah)

- Selective independent families exists under *CH*.
- They are indestructible by a countable support iterations and countable support products of Sacks forcing.

Corollary

It is consistent that $\mathfrak{i} < \mathfrak{c}$.

Countable approximations

Definition (F., Montoya, 2019)

Let $\mathbb{P}$ be the partial order

- of all pairs $(\mathscr{A}, A)$ where $\mathscr{A}$ is a countable independent family and $A \in [\omega]^\omega$ such that for all $h \in \mathrm{FF}(\mathscr{A})$ the set $\mathscr{A}^h \cap A$ is infinite;
- with extension relation defined as follows

$$(\mathscr{B}, B) \leq (\mathscr{A}, A) \text{ iff } \mathscr{B} \supseteq \mathscr{A} \text{ and } B \subseteq^* A.$$

Lemma (CH)

$\mathbb{P}$ is countably closed and $\aleph_2$-cc.

Proof

- Let $\{(\mathscr{A}_i, A_i)\}_{i\in\omega}$ be a decreasing chain in $\mathbb{P}$. Then $\mathscr{A} = \bigcup_{i\in\omega}\mathscr{A}_i$ is a countable independent family.
- Inductively one can construct a pseudointersection A of $\{A_i\}_{i\in\omega}$ such that $A \cap \mathscr{A}^h$ is unbounded for each $h \in \mathrm{FF}(\mathscr{A})$.
- Note that there are only $\aleph_1$ options for a second coordinate and only $\aleph_1$ options for a first coordinate. □

Lemma (CH)

Let G is $\mathbb{P}$-generic. Then $\mathscr{A}_G = \bigcup\{\mathscr{A} : \exists A(\mathscr{A}, A) \in G\}$ is a selective independent family.

Remark

- $\mathscr{A}_G$ is densely maximal;
- $\mathscr{F}_G = \{A : \exists \mathscr{A}(\mathscr{A}, A) \in G\}$ is a Ramsey set;
- $\mathrm{fil}(\mathscr{A}_G)$ is generated by $\mathscr{F}_G$ and so
- $\mathrm{fil}(\mathscr{A})$ is Ramsey filter.

Indestructibility

Let $\mathscr{A}$ be a selective independent family. Then $\mathscr{A}$ remains selective after forcing with the countable support iteration of any of:

- (Shelah, 1989) Shelah's poset for diagonalizing a maximal ideal,
- (Cruz-Chapital, F., Guzman, Supina, 2020) Miller partition forcing,
- (J. Bergfalk, F., C. Switzer, 2021) Coding with perfect trees,
- (Switzer, 2022) *h*-perfect trees,
- (F., Switzer, 2023) Miller lite forcing,

leading in particular to the consistency of each of the following

$$\mathfrak{i} < \mathfrak{u}, \mathfrak{u} = \mathfrak{a} = \mathfrak{i} < \mathfrak{a}_T, \mathfrak{i} = \mathfrak{u} < \mathrm{cof}(\mathscr{N}) = \mathrm{non}(\mathscr{N}), \mathfrak{i} = \mathfrak{hm} < \mathfrak{l}_{n,\omega}.$$

Definition

A poset $\mathbb{P}$ is Cohen preserving if every every new dense open subset of $2^{<\omega}$ (or, equivalently $\omega^{<\omega}$) contains an old dense subset.

Remark

More formally, $\mathbb{P}$ is Cohen preserving if for all $p \in \mathbb{P}$ and all $\mathbb{P}$-names $\dot{D}$ so that

$$p \Vdash \text{“}\dot{D} \subseteq 2^{<\omega} \text{ is dense open”}$$

there is a dense $E \subseteq 2^{<\omega}$ in the ground model, $q \leq_{\mathbb{P}} p$ so that

$$q \Vdash \check{E} \subseteq \dot{D}.$$

Theorem (Shelah)

If δ is an ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \delta, \beta < \delta \rangle$ is a countable support iteration such that for each $\alpha < \delta$

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is proper and Cohen preserving”}$$

then $\mathbb{P}_\delta$ is proper and Cohen preserving.

Lemma

If $\mathbb{P}$ is Cohen preserving and proper, then $\mathbb{P}$ is ω^ω-bounding.

Theorem

Let δ be an ordinal. Let $\mathscr{A}$ be a selective independent family and let $\langle \mathbb{P}_\alpha \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ be a countable support iteration of proper forcing notions so that for every $\alpha < \delta$,

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is Cohen preserving”}.$$

If for every $\alpha < \delta$,

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ preserves the dense maximality of } \mathscr{A}\text{”}$$

then $\mathbb{P}_\delta$ preserves the selectivity of $\mathscr{A}$.

Genericity

Theorem (F., Montoya 2019; F., Switzer 2023)

Let $\mathscr{A}$ be an independent family. Then $\mathscr{A}$ is densely maximal iff $\mathrm{fil}(\mathscr{A})$ is the unique diagonalization filter.

Proof

- (F., Montoya) If $\mathscr{A}$ is densely maximal then $\mathrm{fil}(\mathscr{A})$ is the unique diagonalization filter.
- (F., Switzer) If $\mathrm{fil}(\mathscr{A})$ is the unique diagonalization filter, then $\mathscr{A}$ is densely maximal.

Genericity

Theorem (F., Switzer, 2023)

The generic maximal independent family added by an iteration of Mathias forcing relativized to diagonalization filters is selective.

Genericity

Theorem (F., Switzer, 2023)

(GCH) Let $\kappa < \lambda$ be regular uncountable. It is consistent that

$$\mathfrak{i} = \kappa < \mathfrak{c} = \lambda$$

holds with a selective witness to $\mathfrak{i}$.

The above holds, in fact, for $\kappa < \lambda$ with $\mathrm{cf}(\kappa) > \omega$ and $\mathrm{cf}(\lambda) > \kappa$.

Definition

Let κ be a regular uncountable cardinal, $\mathscr{A} \subseteq [\kappa]^\kappa$.

- Let $\mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ be the set of all finite partial functions with domain included in $\mathscr{A}$ and range the set $\{0,1\}$.
- For each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ let $\mathscr{A}^h = \bigcap\{A^{h(A)} : A \in \mathrm{dom}(h)\}$ where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \kappa \setminus A$ if $h(A) = 1$.

Definition

1. A family $\mathscr{A} \subseteq [\kappa]^\kappa$ is said to be κ-independent if for each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$, $\mathscr{A}^h$ is unbounded.
2. It is maximal κ-independent family if it is κ-independent, maximal under inclusion.
3. The least size of a maximal κ-independent family is denoted $\mathfrak{i}(\kappa)$.

Lemma (F., Montoya)

Let κ be a regular infinite cardinal.

1. There is a maximal κ-independent family of cardinality 2^κ.
2. $\kappa^+ \leq \mathfrak{i}(\kappa) \leq 2^\kappa$
3. $\mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$
4. $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$.

Corollary

If κ is regular uncountable, then if $\mathfrak{i}(\kappa) = \kappa^+$ also $\mathfrak{a}(\kappa) = \kappa^+$.

Definition: κ-dense maximality

A κ-independent family $\mathscr{A}$ is densely maximal if for every $X \in [\kappa]^\kappa \setminus \mathscr{A}$ and every $h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ such that $h' \supseteq h$ and

$$\text{either } \mathscr{A}^{h'} \cap X = \emptyset \text{ or } \mathscr{A}^{h'} \cap (\kappa \setminus X) = \emptyset.$$

Question

Are there κ-densely maximal independent families?

Definition (F., Montoya)

Let κ be a measurable cardinal and $\mathscr{U}$ a normal measure on κ. Let $\mathbb{P}_{\mathscr{U}}$ be the poset of all pairs $(\mathscr{A}, A)$ where

- $\mathscr{A}$ is a κ-independent family of cardinality κ,
- $A \in \mathscr{U}$ is such that $\forall h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$, $\mathscr{A}^h \cap A$ is unbounded.

The extension relation is defined as follows: $(\mathscr{A}_1, A_1) \leq (\mathscr{A}_0, A_0)$ iff $\mathscr{A}_1 \supseteq \mathscr{A}_0$ and $A_1 \subseteq^* A_0$.

Lemma

Assume $2^\kappa = \kappa^+$. Then $\mathbb{P}_{\mathscr{U}}$ is κ^+-closed and κ^{++}-cc.

Proof

- Let $\{(\mathscr{A}_i, A_i)\}_{i\in\kappa}$ be a decreasing sequence in $\mathbb{P}_{\mathscr{U}}$.
- We can assume that $\{A_i\}_{i\in\kappa}$ is strictly decreasing, i.e for each $i > j$ we have $A_j \subseteq A_i$.
- Then $\mathscr{A} = \bigcup_{i\in\kappa} \mathscr{A}_i$ is an independent family of cardinality κ and the diagonal intersection $A' = \Delta_{i\in\kappa} A_i \in \mathscr{U}$.
- Recursively we can define a set A'' which is a pseudo-intersection of $\{A_i\}_{i\in\kappa}$ and which meets every $\mathscr{A}^h$ on an unbounded set.
- Then $A = A' \cup A''$ is an element of $\mathscr{U}$ and so
- $(\mathscr{A}, A) \in \mathbb{P}_{\mathscr{U}}$ is a common extension of $\{(\mathscr{A}_i, A_i)\}_{i\in\kappa}$.

$\mathbb{P}_{\mathscr{U}}$ is κ^{++}-cc, because $|\mathbb{P}_{\mathscr{U}}| = \kappa^+$. □

Lemma (F., Montoya)

Assume $2^\kappa = \kappa^+$, κ is measurable and $\mathscr{U}$ is a normal measure on κ. Let G be a $\mathbb{P}_{\mathscr{U}}$-generic filter. The

$$\mathscr{A}_G = \bigcup\{\mathscr{A} : \exists A \in \mathscr{U} \text{ with } (\mathscr{A}, A) \in G\}$$

is a densely maximal κ-independent family.

Remark: Density filter

Let $\text{fil}_{<\omega,\kappa}(\mathscr{A}_G)$ be the filter of all $X \in \mathscr{U}$ such that $\forall h \in \text{FF}_{<\omega,\kappa}(\mathscr{A}_G)$ there is $h' \in \text{FF}_{<\omega,\kappa}(\mathscr{A}_G)$ such that $h' \supseteq h$ and $\mathscr{A}^{h'} \subseteq X$. Then:

- Every $\mathscr{H} \in [\text{fil}_{<\omega,\kappa}(\mathscr{A}_G)]^{\leq\kappa}$ has a pseudo-intersection in $\text{fil}_{<\omega,\kappa}(\mathscr{A}_G)$.
- If $f \in V \cap {}^\kappa\kappa$ is strictly increasing, then $\exists a \in \text{fil}_{<\omega,\kappa}(\mathscr{A}_G)$ such that

$$f(a(i)) < a(i+1)$$

for all $i \in \kappa$, where $\{a(i)\}_{i\in\kappa}$ is the increasing enumeration of a.

Theorem (F., Montoya)

(GCH) Let κ be a measurable cardinal and let $\mathscr{U}$ be a normal measure on κ. The generic maximal independent family $\mathscr{A}_G$ adjoined by $\mathbb{P}_{\mathscr{U}}$ remains maximal after the κ-support product $\mathbb{S}_\kappa^\lambda$.

Corollary

Let κ be a measurable cardinal. There is a cardinal preserving generic extension in which

$$\mathfrak{a}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{i}(\kappa) = \kappa^+ < 2^\kappa.$$

Thank you for your attention!